ON A CERTAIN PROBLEM IN THE THEORY OF JETS

(OB ODNOI ZADACHE TEORII STRUI)

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V. P. ALEKSEEVSKII
(Kiev)

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This paper deals with the impingement of two streams of an ideal incompressible fluid which move toward each other from infinity in the space bounded by solid walls. The motion is steady.

In the case of plane flow a complete solution of the problem is given by the use of the theory of functions of a complex variable. In the case of axial symmetry certain formulas which are important for practical applications are derived by the use of the law of conservation of momentum

1. Assume that the planes of symmetry of the jet and the bounded flow coincide, the walls are parallel and the space between the walls is filled with the flow to infinity.

Because of the symmetry of such a flow it is necessary to study only a portion of the latter, shown in Fig. 1.

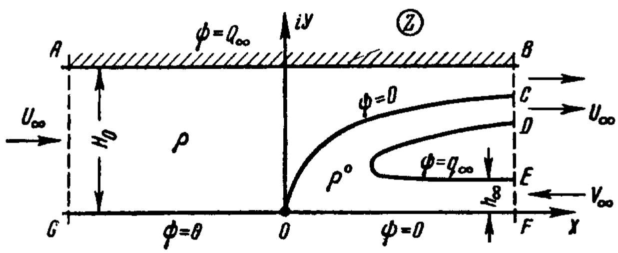

Fig. 1.

Here *AB* is an ideal solid wall, *AG* is the bounded flow, *EF* the jet, *CO* the boundary between the particles of the jet and the flow, *DE* the free surface of the jet, and *GF* the symmetry plane of the flow.

Let us assume that the velocities of the jet and the flow at infinity (cross-sections *AG* and *BF*) do not vary with time.

Under these conditions the motion of the particles of the fluid will be steady with respect to the coordinate system with origin at the point *O*.

As is known, the solution of such a problem is reduced to finding the velocity potential function ϕ which satisfies the Laplace equation $\Delta^2\phi$ and the boundary conditions

$$\frac{\partial\varphi}{\partial n}=0 \quad \text{on} \quad AB \text{ and } GF; \qquad \frac{\partial\varphi}{\partial n}=0, \quad \frac{\partial\varphi}{\partial s}=\text{const} \quad \text{on } DE$$

(*DE* is a free surface), where *n* and *s* are respectively directions of the principal normal and tangent at an arbitrary point of the boundary. Let *W* be a complex potential of the flow:

$$W(z)=\varphi+i\psi, \qquad z=x+iy \tag{1.1}$$

where ψ is the stream function of the flow. Let us find the solution of the given problem by the use of the theory of conformal mapping.

Let u_∞ be the velocity of the bounded flow at infinity, ρ-the density of that fluid, H_0-the half-distance between the solid walls, Q_∞-the strength of the source producing the bounded flow under consideration at infinity, v_∞-the velocity of the jet at infinity, ρ^0-the density of the jet fluid, h_∞-the half-width of the jet at infinity, q_∞-the strength of the source producing the jet under consideration at infinity.

Let us assume at first that $\rho = \rho^0 = 1$. Let us express the boundary conditions of the problem in terms of the stream function ψ. Assuming $\psi = 0$ at the boundary *GOF* and *OC*, we have

$$\psi=Q_\infty \quad \text{on } AB, \qquad \psi=q_\infty \quad \text{on } DE$$

By means of function *W* the region of flow under consideration (Fig. 1) of the *Z*-plane may be mapped conformally into the rectangular region of the *W*-plane shown in Fig. 2. The notations show the location of the boundaries of the region mapped.

Following N.E. Zhukovskii we introduce an auxiliary function

$$Z=\ln\frac{dz}{dW}=X+iY \tag{1.2}$$

The quantity $dW/dz = v_z$ is the complex conjugate of the velocity vector of any given point of the flow.

Separating the real and imaginary parts of *Z*, we write

$$X=\ln|1/v_z|, \qquad Y=\alpha$$

where α is the angle of inclination of the velocity vector with the positive direction of the real axis in the *Z*-plane. Function *Z* brings about the conformal mapping of the region under consideration into the triangular region of the Z-plane.

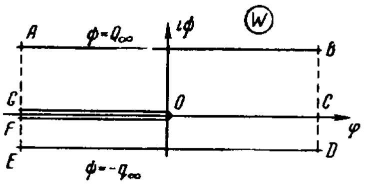

Fig. 2.

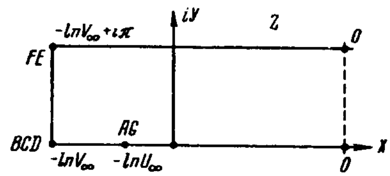

Fig. 3.

The configurations of the region under consideration in the planes W and Z are bounded by straight lines.

When introducing the three points on the ξ-axis which correspond to the three vertices of the rectangular region in the W-plane, as shown in Fig. 4, in the t-plane, on the basis of Christoffel's formula, we obtain

$$W(t) = c_1 \int_0^t t(t-1)^{-1}(t+a)^{-1} dt + t_0 \tag{1.3}$$

Integrating and determining the unknown coefficients from the boundary conditions of the problem, we write

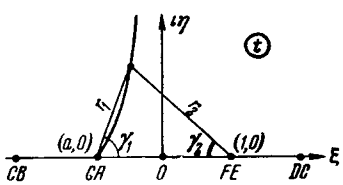

Fig. 4.

$$W(t) = \frac{Q}{\pi} \ln\left[\left(\frac{t+a}{a}\right)^{\frac{a}{a+1}} (1-t)^{\frac{1}{a+1}}\right] \tag{1.4}$$

where

$$Q = Q_\infty + q_\infty, \qquad a = \frac{Q_\infty}{q_\infty} = \frac{u_\infty H_0}{v_\infty h_\infty} \tag{1.5}$$

From (1.4) follows, that

$$\frac{dW}{dt} = \frac{Q}{\pi} \frac{t}{(t+a)(t-1)} \tag{1.6}$$

Analogously, when mapping the upper half-plane (Fig. 4) into the triangular region of the Z-plane (Fig. 3), we find

$$Z(t) = c_2 \int_0^t t^{-1}(t-1)^{-1/2} dt + Z_0$$

Integrating and determining the unknown constants from the boundary conditions of the problem, we obtain

$$Z(t) = -2i\tan^{-1}\sqrt{t-1} - \ln v_\infty + i\pi$$

Using the relationship between the arctangent and logarithm in the complex plane, we transform this expression into the form

$$Z(t) = -\ln\left[v_\infty e^{-i\pi}\frac{1+i\sqrt{t-1}}{1-i\sqrt{t-1}}\right] \tag{1.7}$$

Since $Z(a) = \ln u_\infty$ (Fig. 3) for $t = -a$, when substituting this value in (1.7) and using (1.5), we obtain

$$\frac{u_\infty}{v_\infty} = 1 - 2\sqrt{\frac{h_\infty}{H_0}} \tag{1.8}$$

Eliminating Z from equations (1.2) and (1.7), we write

$$\frac{dW}{dz} = v_\infty e^{-i\pi}\frac{1+i\sqrt{t-1}}{1-i\sqrt{t-1}} \tag{1.9}$$

Dividing (1.9) by (1.6) and integrating the expression obtained with limits from 0 to t, we find

$$z = \frac{Qe^{i\pi}}{\pi v_\infty}\left[-\frac{(1+\sqrt{a+1})^2}{1+a}\ln\left(\frac{t+a}{a}\right) + \frac{1}{a+1}\ln(1-t) + \right.$$
$$\left. + \frac{4\sqrt{a+1}}{1+a}\ln\left(\frac{\sqrt{a+1}-i\sqrt{t-1}}{\sqrt{a+1}+1}\right)\right] \tag{1.10}$$

If it were possible to eliminate the parameter t from (1.4) and (1.10) and then separate the real and imaginary parts of the obtained expression for $W(z)$, we would find the desired solution in an explicit form. However, it is not possible to eliminate t from equations (1.4) and (1.10). Therefore, for their practical use we will find the solution to the problem in parametrical form. We will express complex quantities $t + a$ and $1 - t$ in the following form:

$$t + a = r_1 e^{i\gamma_1}, \qquad 1 - t = r_2 e^{i\gamma_2}$$

where r_1, r_2, γ_1, γ_2 may be taken as bipolar coordinates of the point in the t-plane, connected by the obvious relationships

$$r_1 = (a+1)\frac{\sin\gamma_2}{\sin(\gamma_1+\gamma_2)}, \qquad r_2 = (a+1)\frac{\sin\gamma_1}{\sin(\gamma_1+\gamma_2)} \tag{1.11}$$

Substituting the expressions for $t + a$ and $1 - t$ into (1.4), replacing Q_∞ and q_∞ by their values, and separating the real and imaginary parts of the equations obtained, we write

$$\varphi = \frac{u_\infty H_0}{\pi}\ln\frac{r_1}{a} + \frac{v_\infty h_\infty}{\pi}\ln r_2, \qquad \psi = \frac{u_\infty H_0 \gamma_1}{\pi} - \frac{v_\infty h_\infty \gamma_2}{\pi} \tag{1.12}$$

Analogously, transforming equation (1.10) by the use of the same substitutions, we find

$$x = \frac{H_0}{\pi}\ln\frac{r_1}{a} - \frac{h_\infty}{\pi}\ln r_2 - \frac{H_0}{\pi}(1-n^2)\ln R$$
$$y = \frac{H_0\gamma_1}{\pi} + \frac{h_\infty\gamma_2}{\pi} - \frac{H_0}{\pi}(1-n^2)\gamma \tag{1.13}$$

where

$$R = \sqrt{\frac{n}{a}\left(a + 1 + r_2 + 2\sqrt{r_2(a+1)}\cos\frac{\gamma_2}{2}\right)} \tag{1.14}$$
$$\gamma = -\tan^{-1}\left[\sin\frac{\gamma_2}{2}\left(\sqrt{\frac{r_2}{a+1}} + \cos\frac{\gamma_2}{2}\right)^{-1}\right], \qquad n = 1 - 2\sqrt{\frac{h_\infty}{H_0}}$$

On the basis of (1.13) we find the position of the point D on the free surface at infinity (Fig. 1). We have

$$\gamma_1 = 0, \quad \gamma_2 = \pi, \quad r_1 = r_2 = \infty$$

Hence $\gamma = -1/2\pi$, and the desired quantity

$$y_1 = 2\sqrt{H_0 h_\infty} - h_\infty \tag{1.15}$$

The above formulas (1.12) and (1.13) allow the construction of the stream lines ψ = const and the lines of constant potential ϕ = const in the z-plane of the problem under consideration. Let us determine the velocity of an arbitrary fluid particle v_z by the use of (1.9) and the substitution $1 - t = r_2 \exp(i\gamma_2)$ introduced above.

Denoting the absolute value of the desired quantity by $|v_z| = v_\infty T$ and its argument by $\arg v_z = \beta$, and making the necessary calculations, we obtain

$$T = \sqrt{\frac{r_2 - 2\sqrt{r_2}\cos{}^1/_2\gamma_2 + 1}{r_2 + 2\sqrt{r_2}\cos{}^1/_2\gamma_2 + 1}} \tag{1.16}$$

$$\beta = \tan^{-1}\left[\frac{2\sqrt{r_2}}{1 - r_2}\sin\frac{\gamma_2}{2}\right] \tag{1.17}$$

It is assumed above that $\rho = \rho^0 = 1$. Let us find the condition for which the solution obtained is valid for different densities of the jet and the bounded flow. We will use the fact that the Bernoulli constant

$$C = p_z + \frac{1}{2}\rho|v_z|^2 \tag{1.18}$$

for the flow under consideration is constant at all points of the moving fluid [1]. This condition is fulfilled for different densities of the jet and the bounded flow if, along the boundary separating the two fluids, there is a discontinuity in velocity of the magnitude

$$m = \sqrt{\frac{\rho}{\rho^{\circ}}} \tag{1.19}$$

Assuming that the stream velocity at infinity and the geometry of the flow do not change, by using the relationships (1.8), (1.18) and (1.19) we find the desired condition

$$\frac{u_1}{u_\infty} = \sqrt{\frac{\rho^{\circ}}{\rho}}\left(1 - 2\sqrt{\frac{h}{H}}\right) \tag{1.20}$$

where $h = 2h_\infty$ and $H = 2H_0$ are respectively the width of the jet and the distance between the solid walls, and u_1 is the jet velocity at infinity, for which the exact solution, obtained under the assumption $\rho = \rho^0 = 1$, is valid for any given densities of the jet and the bounded flow.

Consider the motion of the jet and the bounded flow in the coordinate system fixed with respect to the solid walls of the system. The velocity of the jet V at infinity in this coordinate system is obviously determined by the equation $V = v_\infty + u_1$. Also, the velocity u_1 may be regarded as the velocity of penetration of the point O on the boundary separating the jet and the bounded flow (Fig. 1). Using this condition, let us transform (1.20) into the form

$$u_1 = \frac{V}{1 + m/n} = V\left(1 + \frac{\sqrt{\rho/\rho^{\circ}}}{1 - 2\sqrt{h/H}}\right)^{-1} \tag{1.21}$$

This formula determines the magnitude of the velocity of penetration as a function of the absolute jet velocity, the densities of the jet and the bounded flow and the geometry of the flow.

For v_∞ we obtain

$$v_\infty = \frac{V}{1 + n/m} = V\left(1 + \frac{1 - 2\sqrt{h/H}}{\sqrt{\rho/\rho^{\circ}}}\right)^{-1} \tag{1.22}$$

From (1.20) it follows that the depth of penetration l (the distance traversed by the point O in unit of time) is expressed, in terms of the distance traversed in the same time by the particle of the jet situated at infinity L, by the formula:

$$l = L\,\frac{n}{m} = L\,\frac{1 - 2\sqrt{h/H}}{\sqrt{\rho/\rho^{\circ}}} \tag{1.23}$$

The distance between the free surfaces of the divergent stream branches at infinity $h_0 = 2y_1$, is determined in this notation by the following formula:

$$h_0 = 2\sqrt{Hh} - h \tag{1.24}$$

Using (1.8), (1.21), (1.22) and the expression for the absolute velocity at any point of the bounded flow $|v_z|$, we will find the expression for the pressure at any point in the jet or the bounded flow.*

$$p_z = \frac{\rho V^2}{2} \frac{(1 - T^2)}{(m + n)^2} \tag{1.25}$$

In particular, at the point O where, in accordance with (1.16), $T = 0$, we get

$$p_0 = \frac{\rho V^2}{2} \frac{1}{(m + n)^2} \tag{1.26}$$

Note that if the distance between the solid walls is increased indefinitely, i.e. $H_0 \to \infty$, the solution obtained above will become the solution obtained by Lavrent'ev in 1947 (the work has not been published), and also independently by Birkhoff [1], for penetration of the two-dimensional jet into an infinite space filled with incompressible fluid.

2. Now consider the axially-symmetrical case of impingement of the jet and the flow of an ideal weightless fluid in the space bounded by a cylindrical solid wall. This wall is to be at rest. Let us denote the constants at infinity: the velocity and pressure respectively by u_∞^0, p_∞^0 for the jet and by u_∞, p_∞ for the bounded flow.

Let r_∞^0 be the radius of the jet, r_∞ the radius of the free surface, R_∞ the radius of the surface of separation, and R the radius of the solid boundary of the region. We denote the densities of the bounded flow and the jet by ρ and ρ^0 respectively.

Let us find the variation of the momentum dK of the volume of fluid N which is swept through by the rotation of the region $ABCDEFGH$ (Fig. 5) around the z-axis; this volume, in the given coordinate system roz, will assume the configuration $A_1B_1C_1D_1E_1F_1G_1H_1$ after passage of time dt. The quantity dK/dt is determined by the motion of the volumes $N^{(1)}N^{(2)}N^{(3)}N^{(4)}$, the cross-sections of which are shaded in Fig. 5. For the equilibrium process under consideration, this quantity is equal to the sum of hydrodynamic pressures acting on the surface of the volume N under consideration.

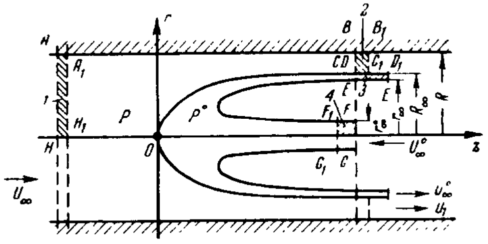

Fig. 5

* It is assumed that there is no external pressure and that $C = 0$ in formula (1.18).

Assuming that the cross-sections AH and BG are sufficiently far from the origin of the coordinate system O and that the corresponding velocities and pressures in these cross-sections are constant and equal to their values at infinity, we write

$$\frac{dK}{dt} = \rho^\circ u_\infty{}^\circ \frac{d}{dt} N^{(3)} + \rho u_0 \frac{d}{dt} N^{(2)} - \rho u_\infty \frac{d}{dt} N^{(1)} + \rho^\circ u_\infty{}^\circ \frac{d}{dt} N^{(3)} = \pi R^2 (p_\infty - p_\infty{}^\circ) \quad (2.1)$$

where u_0 is the flow velocity at cross-section BC, the magnitude of which is determined, as in the two-dimensional case, by the condition of existence of a velocity discontinuity at the boundary of the jet and the bounded flow. It is given by the equation

$$u_0 = u_\infty{}^\circ \sqrt{\rho^\circ / \rho} \quad (2.2)$$

The values of the derivatives of the respective volumes are given by

$$\frac{d}{dt} N^{(1)} = \pi R^2 u_\infty, \qquad \frac{d}{dt} N^{(2)} = \pi (R^2 - R_\infty{}^2) u_0$$
$$\frac{d}{dt} N^{(3)} = \pi (R_\infty^2 - r_\infty{}^2) u_\infty{}^\circ, \qquad \frac{d}{dt} N^{(4)} = -\pi r_\infty{}^{\circ 2} u_\infty{}^\circ \quad (2.3)$$

Writing the Bernoulli equation for an arbitrary streamline which passes through the cross-sections HA and BC, we find

$$p_\infty - p_\infty{}^\circ = \frac{\rho}{2} (u_0{}^2 - u_\infty{}^2) \quad (2.4)$$

Substituting the values of the corresponding quantities (2.2), (2.3) and (2.4) in (2.1), and performing the transformations, we obtain

$$\left[\sqrt{\frac{\rho}{\rho^\circ}} \frac{u_\infty}{u_\infty{}^\circ}\right]^2 = 2 \left(\frac{R^2 - r_\infty{}^2}{R^2} + \frac{r_\infty{}^{\circ 2}}{R^2} - \frac{1}{2}\right) \quad (2.5)$$

Eliminating R from the equations of continuity for the jet and the bounded flow in these cross-sections,

$$u_\infty \pi R^2 = u_0 \pi (R^2 - R^2_\infty), \qquad u_\infty{}^\circ \pi r_\infty{}^{\circ 2} = u_\infty{}^\circ \pi (R_\infty{}^2 - r_\infty{}^2)$$

and using (2.2), we write

$$\left[\sqrt{\frac{\rho}{\rho^\circ}} \frac{u_\infty}{u_\infty{}^\circ}\right]^{\cdot} = \left(\frac{R^2 - r_\infty{}^2}{R^2} - \frac{r_\infty{}^2}{R^2}\right) \quad (2.6)$$

Eliminating r_∞ from (2.5) and (2.6), we obtain the relationship

$$\frac{u_\infty}{u_\infty{}^\circ} = \sqrt{\frac{\rho^\circ}{\rho}} \left(1 - 2 \frac{r_\infty{}^\circ}{R}\right) = \frac{n_0}{m_0} \quad (2.7)$$

which is analogous to (1.20) for the motion in the two-dimensional case. In (2.7)

$$n_0 = \left(1 - 2\frac{r_\infty^\circ}{R_\infty}\right), \qquad m_0 = \sqrt{\rho / \rho^\circ}$$

Assuming $u_\infty + u_\infty^\circ = V$ in the fixed coordinate system (V is the stream velocity, while u_∞ is the velocity of motion of the point O, i.e. the velocity of the boundary of separation of the fluids), we find

$$u_\infty = V\left(1 + \frac{m_0}{n_0}\right)^{-1}, \qquad u_\infty^\circ = V\left(1 + \frac{n_0}{m_0}\right)^{-1} \tag{2.8}$$

From (2.7) and (2.8) there follows that the trajectory l of the point O is expressed in terms of the trajectory L of the jet cross-section by the formula

$$l = L\,\frac{n_0}{m_0} \tag{2.9}$$

From (2.6) and (2.7) it follows that

$$r_\infty = \sqrt{2r_\infty^\circ R - r_\infty^{\circ 2}} \tag{2.10}$$

Assuming that the pressure in the jet at infinity is zero, we find the pressure at the point O:

$$p_0 = \frac{\rho V^2}{2}\,\frac{1}{(m_0 + n_0)^2} \tag{2.11}$$

It should be pointed out that there is at present no exact solution of the three-dimensional problem on the motion of an ideal fluid with free surfaces. Lavrentev's work [2] notes methods of possible solutions of such problems from the mathematical point of view for the case of axial symmetry. Southwell and Vaisey have calculated several cases of motion of the streams with axial symmetry employing the numerical relaxation method.

The relations obtained above permit us, by means of the relaxation method, to calculate approximately the stream lines and equipotential lines of the velocity field and the pressures for the problem under consideration.

In conclusion, let us note that comparison of the analogous expressions determining the parameters of the investigated flow, (1.20), (1.23), (1.24), (1.26) for the case of plane flow, and (2.7), (2.9), (2.10), (2.11) for the case of axial symmetry of motion, shows a substantial difference in the quantitative expression of these parameters.

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